Complexity of Shift Bribery in Hare, Coombs, Baldwin, and Nanson Elections

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Abstract

We show that shift bribery is intractable for the voting systems of Hare, Coombs, Baldwin, and Nanson.

Introduction

One of the main themes in computational social choice is to study the complexity of manipulative attacks on voting systems, in the hope that proving computational hardness of such attacks may provide some sort of protection against them. Besides manipulation (also referred to as strategic voting) and electoral control, much work has been done to study bribery attacks. For a comprehensive overview, we refer to the book chapters by Conitzer and Walsh (2016) for manipulation, by Faliszewski and Rothe (2016) for control and bribery, and by Baumeister and Rothe (2015) for all three topics.

Bribery in voting was introduced by Faliszewski, Hemaspaandra, and Hemaspaandra (2009) (see also Faliszewski et al. (2009)). We will focus on shift bribery, a special case of swap bribery, which was introduced by Faliszewski et al. (2009) in the context of so-called irrational voters for Copeland elections and was then studied in detail by Elkind, Faliszewski, and Slinko (2009). In swap bribery, the briber has to pay for each swap of any two candidates in the votes. Shift bribery additionally requires that swaps always involve the designated candidate that the briber wants to see win.

Swap bribery generalizes the possible winner problem (Konczak and Lang 2005; Xia and Conitzer 2011), which itself is a generalization of unweighted coalitional manipulation. Therefore, each of the many hardness results known for the possible winner problem is directly inherited by the swap bribery problem. This was the motivation for Elkind, Faliszewski, and Slinko (2009) to look at restricted variants of swap bribery such as shift bribery.

Even though shift bribery also possesses a number of hardness results (Elkind, Faliszewski, and Slinko 2009), it has also been shown to allow exact and approximate polynomial-time algorithms in a number of cases (Elkind, Faliszewski, and Slinko 2009; Elkind and Faliszewski 2010; Schlotter, Faliszewski, and Elkind 2011). For example, Elkind, Faliszewski, and Slinko (2009) provided a 2-approximation algorithm for shift bribery when using Borda voting. This result was extended by Elkind and Faliszewski (2010) to all positional scoring rules; they also obtained somewhat weaker approximations for Copeland and maximin voting. For Bucklin and fallback voting, the shift bribery problem is even exactly solvable in polynomial time (Schlotter, Faliszewski, and Elkind 2011).[1] In addition, Bredereck et al. (2014) analyzed shift bribery in terms of *parameterized* complexity.

We study shift bribery for four iterative voting systems that each proceed in rounds, eliminating after each round the candidates performing worst: The system of Baldwin (1926) eliminates the candidates with lowest Borda score and the system of Nanson (1882) eliminates the candidates whose score is lower than the average Borda score, while the system of Hare (defined, e.g., in the book by Taylor (1995)) eliminates the candidates with lowest plurality score and the system of Coombs (defined, e.g., in the paper by Levin and Nalebuff (1995)) eliminates the candidates with lowest veto score. We show NP-completeness of the shift bribery problem for each of these iterative voting systems. Our results complement results by Davies et al. (2014) who have shown unweighted coalitional manipulation to be NP-complete for Baldwin and Nanson voting (even with just a single manipulator)—and also for the underlying Borda system (with two manipulators; for the latter result, see also the paper by Betzler, Niedermeier, and Woeginger (2011)). Davies et al. (2014) also list various appealing features of the systems by Baldwin and Nanson, including that they have been applied in practice (namely, in the State of Michigan in the 1920s, in the University of Melbourne from 1926 through 1982, and in the University of Adelaide since 1968) and that (unlike Borda) they both are Condorcet-consistent.

Preliminaries

Below, we provide the needed notions and notation.

Elections and voting systems. An *election* is specified as a pair (C, V) with C being a set of candidates and V a profile

[1] Faliszewski et al. (2015) have complemented these results on Bucklin and fallback voting. In particular, they studied a number of bribery problems for these rules, including a variant called "extension bribery," which was previously introduced by Baumeister et al. (2012) in the context of campaign management when the voters' ballots are truncated.

of the voters' preferences over C, typically given by a list of linear orders of the candidates. A *voting system* is a function that maps each election (C, V) to a subset of C, the *winner(s) of the election*. An important class of voting systems is the family of positional scoring rules whose most prominent members are plurality, veto, and Borda count, see, e.g., the book chapters by Zwicker (2016) and Baumeister and Rothe (2015). In *plurality*, each voter gives her top-ranked candidate one point; in *veto* (a.k.a. antiplurality), each voter gives all except the bottom-ranked candidate one point; in *Borda* with m candidates, each candidate in position i of the voters' rankings scores $m-i$ points; and the winners in each case are those candidates scoring the most points.

Iterative voting systems. The voting systems we study are based on plurality, veto, and Borda, but their election winner(s) are determined in consecutive rounds. In each round, all candidates with the lowest score are eliminated.[2] If in a round all remaining candidates have the same score (there may be only one candidate left), those candidates are proclaimed winners of the election. We use four different scoring methods: The voting systems due to *Hare*, *Coombs*, and *Baldwin* use, respectively, plurality, veto, and Borda scores, while the *Nanson* system eliminates in every round all candidates that have less than the average Borda score.

Shift bribery. For any given voting system $\mathcal{E}$, we now define the problem $\mathcal{E}$-SHIFT-BRIBERY, which is a special case of $\mathcal{E}$-SWAP-BRIBERY, introduced by Faliszewski et al. (2009) in the context of so-called irrational voters for Copeland and then comprehensively studied by Elkind, Faliszewski, and Slinko (2009). Informally stated, given a profile of votes, a swap-bribery price function exacts a price for each swap of any two candidates in the votes, and in shift bribery only swaps involving the designated candidate are allowed.

$\mathcal{E}$-SHIFT-BRIBERY	
Given:	An election (C, V) with n votes, a designated candidate $p \in C$, a budget B, and a list of price functions $\rho = (\rho_1, \dots, \rho_n)$.
Question:	Is it possible to make p the unique $\mathcal{E}$ winner of the election by shifting p in the votes such that the total price does not exceed B?

Regarding the list of price functions $\rho = (\rho_1, \dots, \rho_n)$ with $\rho_i : \mathbb{N} \to \mathbb{N}$, $\rho_i(k)$ indicates the price the briber has to pay in order to move p in vote i by k positions to the top. For all i, we require that ρ_i is nondecreasing ($\rho_i(\ell) \leq \rho_i(\ell+1)$), $\rho_i(0) = 0$, and if p is at position r in vote i then $\rho_i(\ell) = \rho_i(\ell - 1)$ whenever $\ell \geq r$. The latter condition ensures that p can be shifted upward no farther than to the top.

Notation: If a set $S \subseteq C$ of candidates appears in a vote as $\overrightarrow{S}$, its candidates are placed in this position in lexicographical order. By $\overleftarrow{S}$ we mean the reverse of the lexicographical order of the candidates in S. If S occurs in a vote without an arrow on top, the order in which the candidates from S are placed here does not matter for our argument.

[2]In the original sources stated in the Introduction, certain tie-breaking schemes are used if more than one candidate has the lowest score in some round. For the sake of convenience and uniformity, however, we prefer eliminating them all and disregarding tie-breaking issues in such a case.

Computational complexity. We assume the reader to be familiar with the standard concepts of complexity theory, including the classes P and NP, polynomial-time many-one reducibility, and NP-hardness and -completeness. We will use the following well-known NP-complete problem.

EXACT COVER BY 3-SETS (X3C)	
Given:	Sets $X = \{x_1, \dots, x_{3m}\}$ and $\mathcal{S} = \{S_1, \dots, S_n\}$ such that $S_i \subseteq X$ and $\lvert S_i \rvert = 3$ for all $S_i \in \mathcal{S}$.
Question:	Does there exist an exact cover of X, i.e., a subset $\mathcal{S}' \subseteq \mathcal{S}$ such that $\lvert \mathcal{S}' \rvert = m$ and $\bigcup_{S_i \in \mathcal{S}'} S_i = X$?

We assume that each $x_j \in X$ is contained in exactly three sets $S_i \in \mathcal{S}$; thus $\lvert X \rvert = \lvert \mathcal{S} \rvert$. That X3C even with this restriction is still NP-hard was shown by Gonzalez (1985).

Hare and Coombs

We start by showing hardness of shift bribery for Hare elections.

Theorem 1 *Hare-*SHIFT-BRIBERY *is* NP*-complete.*

Proof. Membership of Hare-SHIFT-BRIBERY in NP is obvious. To prove NP-hardness, we now describe a reduction from X3C to Hare-SHIFT-BRIBERY. Let $(X, \mathcal{S})$ be a given X3C instance with $X = \{x_1, \dots, x_{3m}\}$, $m > 1$, and $\mathcal{S} = \{S_1, \dots, S_{3m}\}$ with $S_i \subseteq X$ and $\lvert S_i \rvert = 3$ for each i, $1 \leq i \leq 3m$. Construct from $(X, \mathcal{S})$ a Hare-SHIFT-BRIBERY instance $((C, V), p, B, \rho)$ as follows. The candidate set is $C = X \cup \mathcal{S} \cup \{p\}$ with designated candidate p, budget B, and list ρ of price functions. For every $S_i \in \mathcal{S}$, let $S_i = \{a_i, b_i, c_i\}$. The list V of votes is constructed as follows (the "$\cdots$" in the table below indicate that the remaining candidates may occur in any order):

#	vote	for
1	$S_i \succ a_i \succ \overrightarrow{X \setminus \{a_i\}} \succ \cdots$	$1 \leq i \leq 3m$
1	$S_i \succ b_i \succ \overrightarrow{X \setminus \{b_i\}} \succ \cdots$	$1 \leq i \leq 3m$
1	$S_i \succ c_i \succ \overrightarrow{X \setminus \{c_i\}} \succ \cdots$	$1 \leq i \leq 3m$
4	$x_i \succ \overrightarrow{X \setminus \{x_i\}} \cdots$	$1 \leq i \leq 3m$
1	$S_i \succ p \succ \cdots$	$1 \leq i \leq 3m$
4	$p \succ \cdots$	

For voters with votes of the form $S_i \succ p \succ \cdots$, we use the price function $\rho'(0) = 0$, $\rho'(1) = 1$, and $\rho'(t) = B + 1$ for all $t \geq 2$; and for every other voter, we use the price function ρ'' with $\rho''(0) = 0$ and $\rho''(t) = B+1$ for all $t \geq 1$. Finally, set the budget $B = m$.

Note that all candidates score exactly four points, so p is not a unique winner without bribing voters.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C, V), p, B, \rho)$ is in Hare-SHIFT-BRIBERY.

($\Rightarrow$) Suppose $(X, \mathcal{S})$ is a yes-instance of X3C. Then there exists an exact cover $\mathcal{S}' \subseteq \mathcal{S}$ of size m. We now show that it is possible for p to become a unique Hare winner of an election obtained by shifting votes without exceeding the budget B. For every $S_i \in \mathcal{S}'$, we bribe the voter with the vote of the form $S_i \succ p \succ \cdots$ by shifting p once, so her new vote is of the form $p \succ S_i \succ \ldots$; each such bribe action costs us only 1 from our budget, so the budget will not be exceeded. In the first round, p now has $m + 4$ points, every candidate from $\mathcal{S}'$ has 3 points, and every other candidate has 4 points. Therefore, all candidates in $\mathcal{S}'$ are eliminated. In the second round, all candidates in X now gain one point from the elimination of $\mathcal{S}'$, since it is an exact cover. Therefore, p and all candidates in X proceed to the next round and the remaining candidates $\mathcal{S} \setminus \mathcal{S}'$ are eliminated. In the next round with only p and the candidates from X remaining, p has $3m+4$ points, while every candidate in X scores 7 points (recall that every $x_i \in X$ is contained in exactly three members of $\mathcal{S}$). Since all candidates from X have been eliminated now, p is the only remaining candidate and thus the unique Hare winner.

($\Leftarrow$) Suppose $(X, \mathcal{S})$ is a no-instance of X3C. Then no subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \leq m$ covers X. We now show that we cannot make p become a unique Hare winner of an election obtained by bribing voters without exceeding budget B. Note that we can only bribe at most m voters with votes of the form $S_i \succ p \succ \cdots$ without exceeding the budget. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that $S_i \in \mathcal{S}'$ exactly if the voter with the vote $S_i \succ p \succ \cdots$ has been bribed. Clearly, $|\mathcal{S}'| \leq m$ and in all those votes p has been shifted once to the left, so she is now ranked first in these votes. Therefore, p now has $4+|\mathcal{S}'|$ points and every $S_i \in \mathcal{S}'$ scores 3 points. Since every other candidate scores the same points as before the bribery (namely, 4 points), the candidates in $\mathcal{S}'$ are eliminated in the first round. Let $X' = \{x_i \in X \mid x_i \notin \bigcup_{S_j \in \mathcal{S}'} S_j\}$ be the subset of candidates $x_i \in X$ that are not covered by $\mathcal{S}'$. We have $X' \neq \emptyset$ (otherwise, $\mathcal{S}'$ would be an exact cover of X). In the second round, unlike the candidates from $X \setminus X'$, the candidates in X' will not gain additional points from eliminating the candidates in $\mathcal{S}'$. Thus, in the current situation, the candidates from X' and $\mathcal{S} \setminus \mathcal{S}'$ are trailing behind with 4 points each and are eliminated in this round. Therefore, in the next round, only p and the candidates from $X \setminus X'$ are remaining in the election. Let $x_\ell \in X \setminus X'$ be the candidate from $X \setminus X'$ with the smallest index. Since all candidates from $\mathcal{S}$ are eliminated, p has $3m + 4$ points and every candidate from $X \setminus X'$ except x_ℓ has 7 points. On the other hand, x_ℓ gains additional points from eliminating the candidates from X'; therefore, x_ℓ survives this round by scoring more than 7 points. In the final round with only p and x_ℓ remaining, p is eliminated, since $3m \cdot 7 > 3m + 4$ for $m > 1$. ❑

Next, we turn to shift bribery for Coombs elections.

Theorem 2 *Coombs-*SHIFT-BRIBERY *is* NP*-complete.*

Proof. Membership of Coombs-SHIFT-BRIBERY in NP is obvious. To prove NP-hardness, we now describe a reduction from X3C to Coombs-SHIFT-BRIBERY. Let $(X, \mathcal{S})$ be a given instance of X3C, where $X = \{x_1, \ldots, x_{3m}\}$ and $\mathcal{S} = \{S_1, \ldots, S_{3m}\}$. Let $S_i = \{x_{i,1}, x_{i,2}, x_{i,3}\}$. We construct an election (C, V) with the set of candidates $C = \{p, w, d_1, d_2, d_3\} \cup X \cup D$, where p is the designated candidate and $D = \{\hat{d}_i \mid x_i \in X\}$. With price functions ρ' and ρ'', defined as $\rho'(0) = 0$, $\rho'(1) = \rho'(2) = \rho'(3) = 1$, and $\rho'(t) = B+1$ for all $t \geq 4$ and $\rho''(0) = 0$ and $\rho''(t) = B+1$ for all $t \geq 1$, we construct the following list V of votes, where from now on we omit the symbol $\succ$ for convenience.

#	vote	for	price fcn.
1	$\cdots\ x_{i,1}\ x_{i,2}\ x_{i,3}\ p$	$1 \leq i \leq 3m$	ρ'
$2m$	$\cdots\ \overrightarrow{D \setminus \{\hat{d}_i\}}\ \hat{d}_i\ x_i$	$1 \leq i \leq 3m$	ρ''
$2m$	$\cdots\ \overrightarrow{D}\ w\ d_1\ d_2\ d_3$		ρ''
1	$\cdots\ \overrightarrow{D}\ w\ X\ d_1\ d_2\ d_3$		ρ''
m	$\cdots\ \overrightarrow{D}\ w$		ρ''

The candidates have the following number of vetoes (denoted by #vetoes):

candidate	#vetoes
p	$3m$
$x_i \in X$	$2m$
w	m
$\hat{d} \in D$	0
d_1	0
d_2	0
d_3	$2m + 1$

Furthermore, our budget is $B = m$.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C, V), p, B, \rho)$ is in Coombs-SHIFT-BRIBERY.

($\Rightarrow$) Assume that (X, S) is in X3C. This means that there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = m$ and $\bigcup_{S_i \in \mathcal{S}'} S_i = X$. So we have a partition of X into three sets, $X = X_1 \cup X_2 \cup X_3$, such that:

$$\begin{aligned} X_1 &= \{x_i \in S_i \in \mathcal{S}' \mid x_i \text{ has the lowest index in } S_i\}, \\ X_3 &= \{x_i \in S_i \in \mathcal{S}' \mid x_i \text{ has the highest index in } S_i\}, \\ X_2 &= X \setminus (X_1 \cup X_3). \end{aligned}$$

Let $D = D_1 \cup D_2 \cup D_3$ be the corresponding partition of D. We bribe the voters with votes of the form $\cdots\ x_{i,1}\ x_{i,2}\ x_{i,3}\ p$ and $S_i \in \mathcal{S}'$ so that they change their votes to $\cdots\ p\ x_{i,1}\ x_{i,2}\ x_{i,3}$. The candidates then have the following number of vetoes in the first round:

candidates	#vetoes
p	$2m$
$x_{i,3} \in X_3$	$2m + 1$
$x \in X \setminus X_3$	$2m$
w	m
$\hat{d} \in D$	0
d_1	0
d_2	0
d_3	$2m + 1$

If a candidate has the highest number of vetoes then she has the fewest number of points and can not go to the next round. Here, the candidates in X_3 and d_3 have the fewest number of points and are eliminated in this round.

Without the candidates in X_3 every candidate in X_2 gets an additional veto and the candidates in D_3 each take the vetoes of the eliminated candidates X_3. This leads to the following number of vetoes for the still-standing candidates in the second round:

candidate	#vetoes
p	$2m$
$x_i \in X_2$	$2m+1$
$x_i \in X_1$	$2m$
w	m
$\hat{d} \in D_3$	$2m$
$\hat{d} \in D \setminus D_3$	0
d_1	0
d_2	$2m+1$

In this round, the candidates in X_2 and d_2 have the fewest number of points and are eliminated. Similarly to the first round, vetoes from candidates in X_2 and d_2 are passed on to candidates in X_1, D_1, and d_1. Thus the candidates receive the following number of vetoes in the third round:

candidates	#vetoes
p	$2m$
$x_i \in X_1$	$2m+1$
w	m
$\hat{d} \in D_2 \cup D_3$	$2m$
$\hat{d} \in D_1$	0
d_1	$2m+1$

All the candidates $x_i \in X_3$ and d_1 are eliminated in this third round, so in the next round there is no candidate $x_i \in X$ and d_i with $1 \leq i \leq 3$. It follows that w receives $2m+1$ additional vetoes in this round, so we have the following number of vetoes in the fourth round:

candidate	#vetoes
p	$3m$
w	$3m+1$
$\hat{d}$	$2m$

So w has the most vetoes in the fourth round and is eliminated. We need $3m$ further rounds until p ends up as the last remaining candidate and sole winner of the election. In each of these rounds, the candidate $\hat{d}$ that is still alive and has the highest index receives at least $2m + 2m + 1 + m$ vetoes, while p always gets only $3m$ vetoes.

($\Leftarrow$) Suppose that (X, S) is a no-instance for X3C. Observe that if we want to make p a unique winner of the election, we have to bribe at least m voters with a vote of the form $\cdots\ x_{i,1}\ x_{i,2}\ x_{i,3}\ p$. If we do not bribe these voters, p has at least $2m + 1$ vetoes and would be eliminated in the first round. Due to our budget we have to bribe exactly m such voters and cannot bribe further voters. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that $S_i \in \mathcal{S}'$ exactly if the voter with the vote of the form $\cdots\ x_{i,1}\ x_{i,2}\ x_{i,3}\ p$ has been bribed. Assume that those voters change their vote from $\cdots\ x_{i,1}\ x_{i,2}\ x_{i,3}\ p$ to $\cdots\ p\ x_{i,1}\ x_{i,2}\ x_{i,3}$ with $S_i \in \mathcal{S}'$. Note that $|\mathcal{S}'| = m$ and $\mathcal{S}'$ does not cover X because we have a no-instance of X3C. Now p has only $2m$ vetoes and will not be eliminated in the first round.

Let X_1 be the candidates $x_i \in S_i$ for $S_i \in \mathcal{S}'$ with the smallest index in S_i. Let X_2 be the candidates $x_i \in S_i$ for $S_i \in \mathcal{S}'$ with the second smallest index in S_i. And let X_3 be the candidates $x_i \in S_i$ for $S_i \in \mathcal{S}'$ with the highest index in S_i. Note that $X_1 \cup X_2 \cup X_3 \neq X$, since $\mathcal{S}'$ does not cover X.

For w to have more vetoes than p, the candidates d_1, d_2, and d_3 need to be eliminated. For that to happen, there must be three rounds in which no other candidate has more than $2m + 1$ vetoes. In the round where d_i, $1 \leq i \leq 3$, is eliminated, all still-standing candidates in X_i are eliminated as well. Assume there were three rounds in which $2m + 1$ was the maximal number of vetoes for a candidate. Then d_1, d_2, d_3, and all candidates in $X_1 \cup X_2 \cup X_3$ are eliminated. Note that those candidates that were not covered by $\mathcal{S}'$ always had only $2m$ vetoes and are still participating in the election. Therefore, in the next round, p and w have $3m$ vetoes each, the remaining candidates from X have at most $2m+1$ vetoes, and the candidates from D have at most $2m$ vetoes. So even if p survives the first rounds with the candidates d_1, d_2, and d_3 still present, she will then surely be eliminated in the following round. ❑

Baldwin and Nanson

We now show NP-hardness of shift bribery for Baldwin and Nanson elections. Note that similar reductions were used by Davies et al. (2014) to show NP-hardness of the unweighted manipulation problem for these election systems.

To conveniently construct votes, for a set of candidates C and $c_1, c_2 \in C$, let

$$W_{(c_1,c_2)} = (c_1\ c_2\ \overrightarrow{C \setminus \{c_1, c_2\}},\ \overleftarrow{C \setminus \{c_1, c_2\}}\ c_1\ c_2).$$

Under Borda, from the two votes in $W_{(c_1,c_2)}$ candidate c_1 scores $|C|$ points, c_2 scores $|C| - 2$ points, and all other candidates score $|C| - 1$ points. Also, observe that if a candidate $c^* \in C$ is eliminated in some round and $c^* \notin \{c_1, c_2\}$ then all other candidates lose one point due to the votes in $W_{(c_1,c_2)}$. If $c^* = c_1$ then c_2 loses no points but all other candidates lose one point, and if $c^* = c_2$ then c_1 loses two points and all other candidates lose one point. Therefore, if c^* is eliminated, the point difference caused by this elimination with respect to the votes in $W_{(c_1,c_2)}$ remains the same for all candidates, with two exceptions: (a) If $c^* = c_1$ then c_2 gains a point from every other candidate, and (b) if $c^* = c_2$ then c_1 loses a point to every other candidate.

Theorem 3 *Baldwin-*SHIFT-BRIBERY *is* NP*-complete.*

Proof. Membership of Baldwin-SHIFT-BRIBERY in NP is obvious. To prove NP-hardness, we reduce the NP-complete problem X3C to Baldwin-SHIFT-BRIBERY. Let $(X, \mathcal{S})$ be a given X3C instance, where $X = \{x_1, \ldots, x_{3m}\}$ and $\mathcal{S} = \{S_1, \ldots, S_{3m}\}$. Let $S_i = \{x_{i,1}, x_{i,2}, x_{i,3}\}$.

We construct an election (C, R) with the set of candidates $C = \{p, w, d\} \cup X \cup \mathcal{S}$, where p is the designated candidate and R consists of two lists of votes, R_1 and R_2. R_1 contains the following votes.

#	votes	for
1	$W_{(S_j,p)}$	$1 \le j \le 3m$
2	$W_{(x_{j,1},S_j)}$	$1 \le j \le 3m$
2	$W_{(x_{j,2},S_j)}$	$1 \le j \le 3m$
2	$W_{(x_{j,3},S_j)}$	$1 \le j \le 3m$
2	$W_{(w,x_i)}$	$1 \le i \le 3m$
7	$W_{(w,p)}$	

For a preference profile P, let $avg(P)$ be the average Borda score of the candidates in P (i.e., $avg(P) = \frac{(|C|-1)|P|}{2}$). Then the votes in R_1 give the following scores to the candidates in C:

$$\begin{aligned} score_{(C,R_1)}(x_i) &= avg(R_1) + 4 \text{ for every } x_i \in X, \\ score_{(C,R_1)}(S_j) &= avg(R_1) - 5 \text{ for every } S_j \in \mathcal{S}, \\ score_{(C,R_1)}(p) &= avg(R_1) - 3m - 7, \\ score_{(C,R_1)}(w) &= avg(R_1) + 6m + 7, \\ score_{(C,R_1)}(d) &= avg(R_1). \end{aligned}$$

Furthermore, R_2 contains the following votes.

#	votes	for
1	$W_{(p,d)}$	
$2m+1$	$W_{(d,S_j)}$	$1 \le j \le 3m$
$2m+9$	$W_{(d,x_i)}$	$1 \le i \le 3m$
$2m+14$	$W_{(d,w)}$	

Then the votes in R_2 give the following scores to the candidates in C:

$$\begin{aligned} score_{(C,R_2)}(x_i) &= avg(R_2) - (2m+9) \text{ for every } x_i \in X, \\ score_{(C,R_2)}(S_j) &= avg(R_2) - (2m+1) \text{ for every } S_j \in \mathcal{S}, \\ score_{(C,R_2)}(p) &= avg(R_2) + 1, \\ score_{(C,R_2)}(w) &= avg(R_2) - (2m+14), \\ score_{(C,R_2)}(d) &= avg(R_2) + 12m^2 + 32m + 13. \end{aligned}$$

Let $R = R_1 \cup R_2$ and $avg(R) = avg(R_1) + avg(R_2)$. Then we have the Borda scores for the complete profile R:

$$\begin{aligned} score_{(C,R)}(x_i) &= avg(R) - 2m - 5 \text{ for every } x_i \in X, \\ score_{(C,R)}(S_j) &= avg(R) - 2m - 6 \text{ for every } S_j \in \mathcal{S}, \\ score_{(C,R)}(p) &= avg(R) - 3m - 6, \\ score_{(C,R)}(w) &= avg(R) + 4m - 7, \\ score_{(C,R)}(d) &= avg(R) + 12m^2 + 32m + 13. \end{aligned}$$

Regarding the price function, for every first vote of $W_{(S_j,p)}$ (i.e., a vote of the form $S_j\ p\ C \setminus \{S_j, p\}$), let $\rho(1) = 1$ and $\rho(t) = m + 1$ for every $t \ge 2$. For every other vote, let $\rho'(t) = m+1$ for every $t \ge 1$. Finally, we set the budget $B = m$. It is easy to see that p is eliminated in the first round in the election (C, R).

We claim that $(X, \mathcal{S})$ is a yes-instance of X3C if and only if $((C, R), p, B, \rho)$ is a yes-instance of Baldwin-SHIFT-BRIBERY.

($\Rightarrow$) Suppose there is an exact cover $S' \subseteq \mathcal{S}$. Then we bribe the first votes of $W_{(S_j,p)}$ for every $S_j \in S'$ by shifting p to the left once. Note that we won't exceed our budget, since shifting once costs 1 in those votes and $|S'| = m$. Additionally, for every $S_j \in S'$, the two votes in $W_{(S_j,p)}$ are now symmetric to each other and can be disregarded from now on, as all candidates gain the same number of points from those votes and all candidates lose the same number of points if a candidate is eliminated from the election. After those m votes have been bribed, only the scores of p and every $S_j \in S'$ change. With $score_{(C,R)}(p) = avg(R) - 2m - 6$ and $score_{(C,R)}(S_j) = avg(R) - 2m - 7$, all candidates in S' are tied for the last place. If any $S_j \in S'$ is eliminated in a round, the three candidates $x_{j,1}$, $x_{j,2}$, and $x_{j,3}$ will lose two points more than the candidates of $S' \setminus \{S_j\}$ that were in the last position before S_j was eliminated. Therefore, those three candidates of X will then be in the last position in the next round. This means that all candidates S' and every $x_i \in X$ that is covered by S' will be eliminated in the subsequent rounds. Since S' is an exact cover, now there is no candidate from X left. Thus the point difference between p and w is 1 and between p and the remaining $S_j \in (\mathcal{S} \setminus S')$ is -6. Note that p can beat d only if no candidate of $C \setminus \{p, d\}$ is still participating. So in the next round, w is eliminated. From this p gains seven points on all $S_j \in (\mathcal{S} \setminus S')$, so these are tied for the last place. Therefore, the remaining candidates from $\mathcal{S}$ are eliminated, which leaves p and d for the next and final round, where d is eliminated and p wins the election alone.

($\Leftarrow$) Suppose there is no exact cover. Then, for every $S' \subseteq \mathcal{S}$ with $|S'| = m$, there is at least one $x_i \in X$ that is not covered by S'. It is obvious that at most m of the first votes of $W_{(S_j,p)}$ can be bribed without exceeding the budget. Without bribing p is on the last place and the point difference to the second-to-last candidate(s) is $dist_{(C,R)}(p, S_j) = m$, $1 \le j \le 3m$. By bribing, as explained above, p gains $m+1$ points on m candidates from $\mathcal{S}$, which then will be eliminated from the election. This leads to the elimination of all $x_i \in X$ that are covered by the set $S' \subseteq \mathcal{S}$ of candidates that were eliminated. Since there is no exact cover, S' doesn't cover X. So there are candidates $X' \subseteq X$, $|X'| \ge 1$, who were not eliminated before, as for every candidate $x_i \in X'$ all three candidates $S_j \in (\mathcal{S} \setminus S')$ with $x_i \in S_j$ are still in the election. With the candidates $C_1 = \{p, w, d\} \cup (\mathcal{S} \setminus S') \cup X'$ still standing, the point differences of p to the other remaining candidates are as follows:

$$\begin{aligned} dist_{(C_1,V)}(p, d) &= -2m + 1 - 2m(2m+1) \\ &\quad -|X'|(2m+9) - (2m+14) < 0, \\ dist_{(C_1,V)}(p, w) &= 1 - 2|X'| < 0, \\ dist_{(C_1,V)}(p, x_i) &= -1 \text{ for every } x_i \in X', \text{ and} \\ dist_{(C_1,V)}(p, S_j) &< -12 \text{ for every} S_j \in \mathcal{S} \setminus S'. \end{aligned}$$

Therefore, p is on the last place and is eliminated. ❑

Finally, we turn to Nanson elections. The reduction below will only use pair of votes $W_{(c_1,c_2)}$. The average Borda score for those two votes is $|C| - 1$. c_1 scores one point more than the average and c_2 scores one point fewer than the average. The other candidates score exactly the average Borda score.

If a candidate is eliminated in a round, the average Borda score required to survive the next round decreases by one. Regardless of which candidate is eliminated, all remaining candidates that are not c_1 or c_2 lose one point and still have exactly the average Borda score. If c_2 is eliminated, c_1 loses its advantage over the average and now scores exactly the average Borda score as well. If one of the other candidates is eliminated, c_1 continues to have one point more than the average Borda score. This is analogous for c_2: If c_1 is eliminated, c_2 scores the average Borda score, and if one of the other candidates is eliminated, c_2 still has one point fewer than the average Borda score.

Theorem 4 *Nanson*-SHIFT-BRIBERY *is* NP*-complete.*

Proof. Membership of Nanson-SHIFT-BRIBERY in NP is obvious. To prove NP-hardness, we reduce the NP-complete problem X3C to Nanson-SHIFT-BRIBERY. Let $(X, \mathcal{S})$ be a given X3C instance, where $X = \{x_1, \ldots, x_{3m}\}$ and $\mathcal{S} = \{S_1, \ldots, S_{3m}\}$. Let $S_i = \{x_{i,1}, x_{i,2}, x_{i,3}\}$.

We construct an election (C, R) with the set of candidates $C = \{p, w_1, w_2, d\} \cup X \cup \mathcal{S}$, where p is the designated candidate. Then we construct two sets of votes, R_1 and R_2. R_1 contains the following votes.

#	votes	for
1	$W_{(S_j,p)}$	$1 \leq j \leq 3m$
1	$W_{(x_i,p)}$	$1 \leq i \leq 3m$
1	$W_{(x_{j,1},S_j)}$	$1 \leq j \leq 3m$
1	$W_{(x_{j,2},S_j)}$	$1 \leq j \leq 3m$
1	$W_{(x_{j,3},S_j)}$	$1 \leq j \leq 3m$
4	$W_{(S_j,w_1)}$	$1 \leq j \leq 3m$
$15m$	$W_{(w_1,w_2)}$	
$3m$	$W_{(p,w_1)}$	

Furthermore, R_2 contains the following votes.

#	votes	for
$2m$	$W_{(p,d)}$	
2	$W_{(d,S_j)}$	$1 \leq j \leq 3m$
4	$W_{(d,x_i)}$	$1 \leq i \leq 3m$

Let $R = R_1 \cup R_2$. Then we have the following Borda scores for the complete profile R:

$$\begin{aligned} score_{(C,R)}(x_i) &= avg(R) \text{ for every } x_i \in X, \\ score_{(C,R)}(S_j) &= avg(R) \text{ for every } S_j \in \mathcal{S}, \\ score_{(C,R)}(p) &= avg(R) - m, \\ score_{(C,R)}(w_1) &= avg(R), \\ score_{(C,R)}(w_2) &= avg(R) - 15m, \\ score_{(C,R)}(d) &= avg(R) + 16m. \end{aligned}$$

The price function is again defined as follows. For every first vote of $W_{(S_j,p)}$ (i.e., a vote of the form $S_j\, p\, C \setminus \{S_j, p\}$), let $\rho(1) = 1$ and $\rho(t) = m + 1$ for every $t \geq 2$. For every other vote, let $\rho'(t) = m + 1$ for every $t \geq 1$. Finally, we set the budget $B = m$. It is easy to see that p is eliminated in the first round in the election (C, R).

We claim that $(X, \mathcal{S})$ is a yes-instance of X3C if and only if $((C, R), p, B, \rho)$ is a yes-instance of Nanson-SHIFT-BRIBERY.

($\Rightarrow$) Suppose there is an exact cover $S' \subseteq \mathcal{S}$. Then, for every $S_j \in S'$, we bribe the first vote of $W_{(S_j,p)}$ by shifting p to the left once in all those votes. Note that we won't exceed our budget, since this bribe action costs 1 per vote and $|S'| = m$. With the additional m points p reaches the average Borda score and is not eliminated in the first round. However, all candidates in S' lose one point and are eliminated. Additionally, w_2 will be eliminated as well.

In the next round, w_1 will be eliminated, since she has $11m$ points fewer than the average score required to survive this round. Since the candidates in S' were eliminated in the last round and S' is an exact cover, every candidate in X now has fewer points than the average and is eliminated.

In the third round, only p, d, and the candidates in $\mathcal{S} \setminus S'$ are still standing. Therefore, the only pairs of votes that are not symmetric are $W_{(S_j,p)}$, twice $W_{(d,S_j)}$ for every $S_j \in (\mathcal{S} \setminus S')$, and $2m$ pairs of $W_{(p,d)}$. Since $|\mathcal{S} \setminus S'| = 2m$, we have that p scores exactly the average score and survives this round, just as d. Every $S_j \in (\mathcal{S} \setminus S')$ has one point fewer than the average and is eliminated. This leaves only p and d in the last round, which p wins.

($\Leftarrow$) Suppose there is no exact cover. Then, for every $S' \subseteq \mathcal{S}$ with $|S'| = m$, there is at least one $x_i \in X$ that is not covered by S'. Note that we can only bribe the first votes of any $W_{(S_j,p)}$ without exceeding the budget. For p to survive the first round, we need to bribe m of those votes by shifting p to the left once. Let $S' \subseteq \mathcal{S}$ be such that S' contains S_j exactly if the first vote of $W_{(S_j,p)}$ has been bribed. Then every $S_j \in S'$ has a score of $avg(R) - 1$ and p has a score of $avg(R)$. Therefore, in the first round, every candidate from S' and w_2 are eliminated from the election.

In the second round, w_1 will be eliminated because of the $15m$ pairs of votes $W_{(w_1,w_2)}$ and the elimination of w_2. Furthermore, a candidate $x_i \in X$ reaches the average score with p and d still standing only if all three $S_j \in \mathcal{S}$ with $x_i \in S_j$ are also not yet eliminated. Since the candidates in S' were eliminated in the previous round, for every $S_j \in S'$, all three $x_i \in S_j$ will be eliminated in this round. Since S' is not an exact cover, there are candidates $X' \subseteq X$ that survive this round. d also reaches the average, as there are $2m$ candidates $\mathcal{S} \setminus S'$ and those candidates $\mathcal{S} \setminus S'$ survive because of w_1.

In the next round, the candidates still standing are p, d, X', and $\mathcal{S} \setminus S'$. Because $|X'| \geq 1$, candidate p has $|X'|$ points fewer than the average score and is eliminated in this round. ❑

Conclusions and Open Questions

We have shown that shift bribery is NP-complete for each of the iterative voting systems of Hare, Coombs, Baldwin, and Nanson. While these are interesting theoretical results complementing earlier work both on shift bribery and on these voting systems, NP-hardness of course has its limitations in terms of providing protection in practice. It would be interesting to also study shift bribery for these systems in terms of approximation and parameterized complexity.

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